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CENTRIFUGAL WAVES IN A PROGRESSIVELY ROTATING FLUID FLOW*

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It is shown that it is possible for solitons to arise in the progressively rotating flow of an ideal incompressible fluid. Such a motion is characteristic for vortical MHD generators and small-scale atmospheric vortices.

1. Equations of motion and boundary conditions. Let a progressively rotational fluid flow be created in a rigid tube with internal radius R by means of a tangential input and pressure drop. This leads to the formation in the tube of a cylindrical cavity of radius r_0 filled with air or, if the tube does not communicate with the atmosphere, the saturated vapour of the fluid (Fig.1). Any perturbation $\eta(z, t)$ of the radius of the cavity may propagate along the axis of the tube in the form of plane waves. In future, we shall assume that the maximum amplitude of the perturbation, a, is small compared with h, the thickness of the fluid layer, and l, the length of the perturbation is, in the other hand, large compared with h. This leads to the following parameters being small: $\varepsilon = a/h$ and $\delta = h/l$. Here the thickness of the fluid layer is taken to be small compared with the radius of the tube, so that $h = (R^2 - r_0^2)/(2r_0)$.

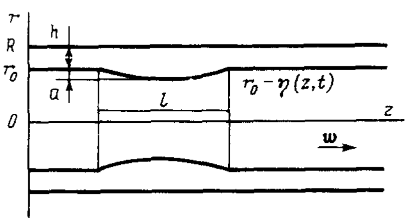

Fig.1

Because of the axial symmetry of the fluid boundary (but not of the flow), in a cylindrical coordinate system the components of $\mathbf{v}$, the velocity vector of the fluid, depend only on the distance r from the flow axis, the z coordinate and t, the time. The vorticity ω of the flow is taken to be constant along the tube and directed along the z axis, so that there is no angular dependence.

Assuming that the fluid is incompressible, we can introduce the vector potential $\mathbf{A}$ such that $\mathbf{v} = \operatorname{rot} \mathbf{A}$, and reduce the problem to solving Poisson's equation $\Delta \mathbf{A} = -\boldsymbol{\omega}$.

On the free surface of the fluid $r_1 = r_0 - \eta(z, t)$ (here and below the index 1 refers to quantities that are calculated on the free surface) the kinematic boundary condition can be written in the form

$$v_{r1} = -\left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial z} v_{z1}\right) \tag{1.1}$$

The dynamic boundary condition is obtained from the Euler equation by substituting expressions for the pressure in the rotating fluid at an arbitrary point of the free surface into it

$$p_1 = p_0 + {}^1/_2 \rho M^2 (r_0^{-2} - r_1^{-2})$$

where $M = v_\varphi r_1$ is the constant specific angular momentum of the fluid. The variation in the fluid pressure on the free surface caused by its perturbation (with $r_1 \approx r_0$) is equal to $dp_1 = \rho v_\varphi^2 r_0^{-1} d\eta$. Consequently, the radial and azimuthal projections of the Euler equation express the constant nature of the radial and azimuthal components of the fluid flow velocity on the boundary with the gas vortex, and the axial, projection gives the dynamical boundary condition:

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$$\frac{\partial v_{z1}}{\partial t} + v_{r1}\frac{\partial v_{z1}}{\partial r} + v_{z1}\frac{\partial v_{z1}}{\partial z} + \frac{v_\varphi^2}{r_0}\frac{\partial \eta}{\partial z} = 0 \quad (1.2)$$

The v_{r1} and v_{z1} that occur in the boundary conditions (1.1) and (1.2) are determined by the component A_φ of the potential, which obeys the Laplace equation. Bearing in mind the thinness of the fluid layer, we can write the solution to this equation in the form

$$A_\varphi(r, z, t) = \sum_{n=0}^{\infty} (R^2 - r^2)^n A_{\varphi n}(z, t) \quad (1.3)$$

Substituting (1.3) into the Laplace equation and separating the terms according to the powers of $(R^2 - r^2)$, we obtain a recurrence relation

$$A_{\varphi n+1} = \frac{1}{4(n+1)}\frac{\partial^2 A_{\varphi n}}{\partial z^2}\left(1 - \frac{nr^2}{R^2 - r^2}\right)^{-1} \quad (1.4)$$

It follows from the conditions on the wall of the tube that $\partial A_{\varphi 0}/\partial z = 0$, which cannot be said about the higher-order derivatives. Thus, according to (1.3) we have the following on the free fluid surface:

$$A_{\varphi 1} = A_{\varphi 0} + {}^1/_4(R^2 - r_1^2)f + {}^1/_{32}(R^2 - r_1^2)^2\left(1 - \frac{r_1^2}{R^2 - r_1^2}\right)^{-1}\frac{\partial^2 f}{\partial z^2} + \ldots \quad (1.5)$$
$$f(z, t) = \partial^2 A_{\varphi 0}(z, t)/\partial z^2$$

2. The linear approximation. We linearize the boundary conditions (1.1), (1.2), and omit the terms that are quadratic in the variable quantities:

$$v_{r1} = -\frac{\partial \eta}{\partial t}, \quad \frac{\partial v_{z1}}{\partial t} = -\frac{v_\varphi^2}{r_0}\frac{\partial \eta}{\partial z}$$

Substituting the first terms of the expansions for v_{r1} and v_{z1} according to (1.5) into the left-hand sides of these expressions and then eliminating f, we find that the radial mixing η of the free fluid surface obeys a linear wave equation:

$$\partial^2\eta/\partial t^2 - c_0^2\partial^2\eta/\partial z^2 = 0, \quad c_0 = r_0^{-1}v_\varphi\sqrt{(R^2 - r_0^2)/2}$$

where c_0 is the velocity of the non-dispersive centrifugal wave of the form

$$\eta = a\exp[i(\omega t \mp kz)], \quad \omega = c_0 k$$

that propagates over the surface of the progressively rotating flow.

The result we have obtained is in complete agreement with the conclusion reached in /1/.

3. Centrifugal solitons. Introducing the dimensionless variables

$$z' = z/l, \quad t' = c_0 t/l, \quad \eta' = \eta/a, \quad v_{z1}' = v_{z1}/(\varepsilon c_0)$$
$$v_{r1}' = v_{r1}/(\varepsilon\delta c_0), \quad f' = r_0 f/(2\varepsilon c_0), \quad A_{\varphi 0}' = r_0 A_{\varphi 0}/(2\varepsilon c_0 l^2)$$

and setting $R^2 - r_1^2 = (R^2 - r_0^2)(1 + \varepsilon\eta')$, we can rewrite the boundary conditions (1.1) and (1.2), up to the first non-vanishing order in ε and δ, as

$$\frac{\partial \eta'}{\partial t'} - \frac{\partial f'}{\partial z'} - \varepsilon\eta'\frac{\partial f'}{\partial z'} - \varepsilon f'\frac{\partial \eta'}{\partial z'} + {}^1/_2\delta^2\frac{\partial^3 f'}{\partial (z')^3} = 0 \quad (3.1)$$

$$-\frac{\partial f'}{\partial t'} + {}^3/_2\delta^2\frac{\partial^3 f'}{\partial (z')^2\partial t'} + \frac{\partial \eta'}{\partial z'} + \varepsilon f'\frac{\partial f'}{\partial z'} = 0 \quad (3.2)$$

We can find η' from (3.1) and (3.2) by the method of perturbations /2/. To this end, we introduce an expansion of f' in the small parameters $f' = \eta' + \varepsilon f^{(1)} + \delta^2 f^{(2)}$ into them, and keep the terms up to the first order in ε and δ^2. Then, if we put together the equations so formed and take into account that $\partial f^{(n)}/\partial t' = \partial f^{(n)}/\partial z'$ $(n = 1, 2)$ to the required accuracy, we obtain the following expression:

$$\varepsilon\left(\frac{\partial f^{(1)}}{\partial z'} + \eta'\frac{\partial \eta'}{\partial z'}\right) + \delta^2\left(\frac{\partial f^{(2)}}{\partial z'} - \frac{\partial^3 f'}{\partial (z')^3}\right) = 0$$

Since the parameters ε and δ are independent, this coefficients must be equal to zero. Taking account of this, we rewrite (3.1) in the form

$$\frac{\partial \eta'}{\partial t'} - \frac{\partial \eta'}{\partial z'} - \varepsilon\eta'\frac{\partial \eta'}{\partial z'} - {}^1/_2\delta^2\frac{\partial^3 \eta'}{\partial (z')^3} = 0 \quad (3.3)$$

Returning to dimensional variables and using the transformation $\xi = z + c_0 t,\ \tau = -\varepsilon t$, we can reduce (3.3) to the Korteveg-de Vries equation

$$\eta_\tau + c_0 h^{-1} \eta\eta_\xi - {}^1/_2 c_0 h^2 \eta_{\xi\xi\xi} = 0 \tag{3.4}$$

the solution of which has the following form in z, t variables:

$$\eta = \eta_0 \operatorname{sech}^2 \frac{z + Vt}{L}, \quad L = 2 \sqrt{\frac{3h^3}{2\eta_0}}, \quad V = c_0\left(1 + \frac{1}{3}\frac{\eta_0}{h}\right) \tag{3.5}$$

where η_0 is the amplitude, determined by the initial conditions, of the soliton, L is its width and V is the velocity of propagation.

Thus, in progressively rotating fluid flow, the centrifugal waves propagating on its free surface may take the form of solitons at a fairly large oscillation amplitude. These solitons are similar to the solitons in shallow water in the case of a plane fluid surface /3/, but they have a number of peculiarities that are associated with the rotation of the fluid. An important property of the centrifugal solitons, as can be seen from (3.5) and the assumption about the direction of the vorticity of the flow (it determines the sign of the angular component of the vector potential), is its left-hand screw character: the directions of the angular momentum and of the soliton velocity are opposite to each other.

Eq. (3.4) can be obtained from the Boussinesque equation

$$\eta_{tt} = c_0^2 (\eta + h^{-1}\eta^2 + h^2\eta_{zz})_{zz}$$

using the change of variables $\xi = z + c_0 t,\ \ \tau = -\varepsilon t\ (\varepsilon \ll 1)$ and discarding terms of order ε^2. This admits of a solution in the form of two solitary waves that are described by the equation

$$\eta = \eta_0 \operatorname{sech}^2 [\sqrt{{}^2/_3 \eta_0 h^{-3}}\,(z \pm ct)]$$

and propagate towards each other along the z axis with velocity $c = c_0\sqrt{1 + {}^2/_3 \eta_0 h^{-1}}$, where one of the waves is left-handed and the other is right-handed. The transformation $t = -\tau/\varepsilon$ signifies a time delay and is equivalent to assuming that the velocity of the forward motion of the liquid is small. An increase in this velocity, as is clear from the exposition above, is associated with the transition to a regime of preferential excitation of the left handed solitary wave described by the Korteveg-de Vries equation which propagates in the direction of the fluid flow in the case of a left-handed flow and in the opposite direction in the case of a right-handed flow.

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